Scalable Graphical Framework for Fermionic Computing: From premonoidality in the semantics to non-natural symmetry in the diagrammatic syntax

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Abstract

Fermionic quantum computing offers an alternative to the standard qubit-based model, in which information is encoded in the occupancy of fermionic modes possibly containing indistinguishable particles that obey Fermi-Dirac statistics. The algebraic structure of such systems, captured by Canonical Anticommutation Relations (CAR) algebras, naturally leads to a premonoidal categorical semantics, reflecting the failure of the exchange law due to the antisymmetry of fermions. Permutation of two fermionic modes in Bravyi-Kitaev's model can be performed through the fermionic swap, which makes this categorical semantics a symmetric premonoidal category. While this structure corresponds to the underlying denotational semantics of Bravyi and Kitaev's early computational model [**?**], it is ill-suited for circuit manipulation, as premonoidal categories lack the exchange law enabling intuitive diagrammatic reasoning.
In this work, we address this challenge by embedding the symmetric premonoidal category of fermionic processes into a presymmetric monoidal category via a functor defined using the Jordan-Wigner mapping. Interestingly, this functor does not map the fermionic swap to a symmetry for the Qubit category equipped with the usual tensor product, due to a default of naturality. This allows us to recover a setting where diagrammatic syntax is more reasonable—the categories involved are monoidal categories, and more specifically props, equipped with a

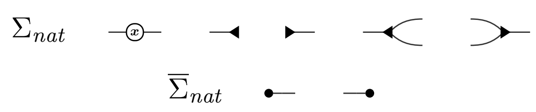
non-natural symmetry given by the image of the fermionic swap. We provide axiomatizations for these categories.
To do so, we build on the work of De Felice et al. [?], who provided prop theories for the same categories seen as symmetric monoidal with respect to the classical swap. We argue that the use of the classical swap as the canonical symmetry of a prop, or as a generator at all, is unsatisfactory in the fermionic setting, as it obscures diagrammatic reasoning and complicates the data structures needed to automatize this reasoning.
Our contribution is threefold:

- **Categorical Foundations:** We define the CAR category and its presymmetric monoidal structure, establishing the first rigorous categorical semantics based on algebraic quantum mechanics for Bravyi-Kitaev's fermionic computational model. This framework reveals deep connections to effectful computations for non-commutative monads, as premonoidal categories arise naturally in the Kleisli categories of such monads.

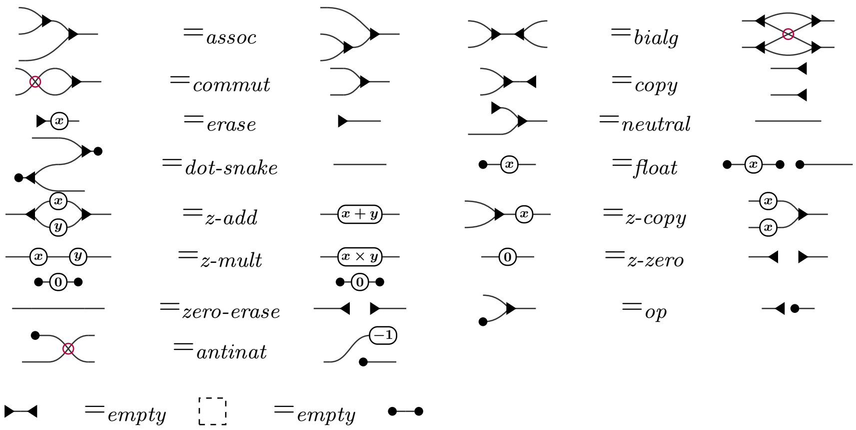
- **Diagrammatic Syntax for Fermionic Circuits:** We introduce pronnaps (PROducts and Non-NAtural Permutations), a generalization of props to presymmetric monoidal categories, as the syntactic counterpart for fermionic circuits. Building on this, we develop complete diagrammatic theories for fragments of fermionic computation, including:

  - FZWbis, EZWbis, which are pronnap theories with the fermionic swap as canonical presymmetry, whose generators and equations are deduced from FZW and EZW from [?].
  - pWbis, which is an alternative pronnap theory with the fermionic swap as canonical presymmetry to the prop theory pW encapsulating generalised matchgates that was developed in [?].
  - EpW, EaW, aW which are structurally crucial fragments of pW, associated with subsets of Bravyi and Kitaev's universal gate set for fermionic computing.

  Example. *The aW pronnap theory has generators :*

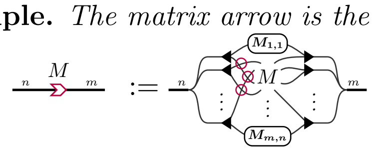
  and Equations :



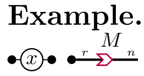
- **Scalable Notations and Normal Forms:** To handle the combinatorial explosion of large fermionic circuits, we extend scalable notations (originally designed for symmetric monoidal categories in [**?**]) to the presymmetric setting. This enables us to define:

  – Matrix arrows and graph nodes: Fermionic analogs of Clifford computation tools revealing further structural similarities between matchgates, i.e. fermionic linear optics, and Clifford circuits.

  Example. *The matrix arrow is the following syntactic sugar in SaW :*



  – Normal forms: A new AP-normal-form-like structure for fragments aW and pWbis, providing elegant proofs of completeness and paving the way for Clifford-inspired optimization algorithms.

  Example. *The normal form for SaW using matrix arrows is*  *with M a full rank column reduced echelon form matrix, $x \in \mathbb{C}$.*

References

[1] Bravyi, S., & Kitaev, A. (2002). Fermionic quantum computation.

[2] de Felice, M., Hadzihasanović, A., & Ng, N. (2019). A diagrammatic calculus of fermionic quantum circuits.

[3] Carette, T., Moutot, E., Perez, T., & Vilmart, R. (2023). Compositionality of planar perfect matchings.

[4] Carette, T., Horsman, D., and Perdrix, S. (2019) SZX-Calculus: Scalable Graphical Quantum Reasoning.