Remarks on proof nets as combinatorial maps

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Abstract

This is a proposal for a talk concerning some technically easy yet little-known observations. I previously presented this material briefly in a talk at Università Roma Tre in May 2019, but it is not written down anywhere other than my slides. My apologies for the paucity of drawings in this hastily prepared abstract.

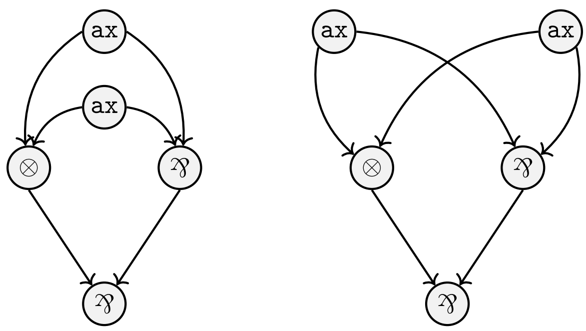
The key concept is that of *combinatorial map*: discrete objects that correspond to topological embeddings of graphs on surfaces. Since the 2010s, they have made prominent appearances in the bijective and enumerative combinatorics of linear λ-terms.[1] However, they are virtually never mentioned in conjunction with proof nets. My contention is that combinatorial maps:

- provide the right notion of *planarity* for the proof nets of non-commutative (cyclic) MLL;
- explain what is happening in Melliès's "ribbon"[2] correctness criterion for cyclic MLL;
- shed some light on the historical "long trip" correctness criterion, both in the commutative and non-commutative cases.

Cyclic MLL. In the sequent calculus for cyclic MLL, the exchange rule (below, left) is replaced by cyclic exchange (right):

$$\frac{\vdash \Gamma, A, B, \Delta}{\vdash \Gamma, B, A, \Delta} \qquad \text{vs} \qquad \frac{\vdash \Gamma, \Delta}{\vdash \Delta, \Gamma}$$

This translates to proof nets "drawn on the plane without crossings": of the two MLL proof nets below, only the left one is correct for cyclic MLL.



[1] cf. `https://www.lix.polytechnique.fr/LambdaComb/` — I would like to thank the LambdaComb community, as well as Éric Colin de Verdière, for discussions on combinatorial maps.

[2] *A topological correctness criterion for multiplicative non commutative logic*, 2004

But these two proof nets are equal as vertex-labeled directed graphs… Thus the notion of *planar graph* is insufficient. This has of course been noticed in the literature: let me quote for instance Nagayama & Okada (emphasis mine):[3]

> "A marked D–R graph drawing is said to be uniformly directed if the L-edge, R-edge and C-edge for a link is *drawn in a fixed cyclic order uniformly* for all tensor-links and par-links, or the links of degree 3."

In other words, they ask us to consider graphs endowed with a *rotation system*: for each vertex, a cyclic order on its incident edges. (This order is different for our two examples.)

Definition 1. A *combinatorial map* is an undirected graph endowed with a rotation system.

Theorem 2 (Heffter–Edmonds–Ringel principle)**.** *Connected combinatorial maps correspond bijectively to homeomorphism classes of embeddings of graphs on surfaces. More precisely:*

- *The surfaces are compact oriented manifolds without boundary.*
- *The embeddings are cellular, i.e. the faces are homeomorphic to disks.*

Definition 3. A map is *planar* if the surface is a sphere (the 1-point compactification of the plane).

Planarity is determined by combinatorial data: the rotation system determines the *faces* of a combinatorial map, and therefore its Euler characteristic, which can be computed in linear time.

A correctness criterion for cyMLL. For *cut-free* proof structures,

$$\text{MLL correctness} + \text{planarity} = \text{cyclic MLL correctness}.$$

For proof nets with cuts, things are more subtle. Melliès (2004) proposes a criterion based on "ribbons": proof nets are represented as some kind of thickened graphs.

I claim that this point of view amounts to the same thing as a graph embedded on a surface. Indeed, for small enough $\varepsilon > 0$, taking the ϵ-neighborhood in the surface of the graph results in the corresponding ribbon. Conversely, one can recover the faces of a combinatorial map from the connected components of the boundary of its ribbon.

This allows me to rephrase the ribbon correctness criterion:

Theorem 4 ($\approx$ Melliès 2004)**.** *A proof structure with cuts is a cyMLL proof net iff*

- *it is a MLL proof net and a planar map,*
- *all its conclusions are on the same face, and this face contains no upper corner of a* $\parr$*-link.*

Corollary 5. *Correctness for cyclic MLL proof nets with cuts is decidable in linear time.*

I do not think this statement appears in the literature. At least, in a paper from 2019, Abrusci and Maieli[4] appear to be unaware of it: they merely cite a quadratic-time algorithm for the cut-free case.[5] Perhaps this is due to the fact that ribbons do not appear to be *prima facie* computationally convenient objects. Here, the combinatorial map point of view proves useful.

[3] *A graph-theoretic characterization theorem for multiplicative fragment of non-commutative linear logic*, 2003, Definition 3.2.
[4] *Proof nets for multiplicative cyclic linear logic and Lambek calculus*
[5] Mogbil, *Quadratic Correctness Criterion for Non-commutative Logic*, 2001

Long trip switchings as embedded graphs. Girard's original *long trip* correctness criterion for MLL (the usual, commutative one) is as follows:

- On each $\otimes$-link and $\parr$-link, choose 1 of 2 possible sets of routing instructions around it.
- Is the orbit a single cycle for all choices?

It is folklore that a long trip specifies a way to travel around a Danos–Regnier switching of a proof structure. But how should we interpret the choices of routing around a $\otimes$-link? The map-pilled answer: it's a choice of *rotation system*, and thus trips are *faces*.

The equivalence of the long trip and Danos–Regnier correctness criteria is thus an instance of:

Theorem 6. *A connected graph is a tree iff all combinatorial maps with this underlying graph have a single face.*

Via the Heffter–Edmonds–Ringel principle, one can rephrase the condition topologically: all embeddings of the graph on surfaces have a single face. Put this way, it is visually intuitive: the boundary of a non-trivial face is a cycle.

(During his PhD, Seiller has proved a "correctness criterion for trees" which amounts to this theorem once the combinatorial definitions are unfolded.[6])

Non-commutative long trips. The connection between trips and combinatorial maps explains in hindsight why Abrusci & Ruet[7] were successful in adapting the long trip correctness criterion to cyclic MLL, without explicitly mentioning planarity.

In *The Blind Spot* (§11.3.6), Girard writes that the long trip approach "remains irreplaceable in the case of non-commutative logic". The material I have just exposed is an argument against this assertion.

[6] `https://www.seiller.org/documents/articles/CCTrees.pdf`

[7] *Non-commutative logic I: the multiplicative fragment*, 2000